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Note

Comment on "Comparison of Some Methods for Evaluating Infinite Range Oscillatory Integrals"

In their recent paper, "Comparison of some methods for evaluating infinite range oscillatory integrals," Blakemore *et al.* [1] have omitted mention of a method which, when it is applicable, can be greatly superior to all the methods they discuss: namely, deformation of the path of integration into the complex plane.

Their integrals are of the form

$$I(\omega) = \int_0^\infty dx\, f(x)\, W(x), \tag{1}$$

where $W(x)$ is an oscillatory function, such as $\sin \omega x$, $\cos \omega x$ or a Bessel function, $J_\nu(\omega x)$. Suppose for example $W(x) = \cos \omega x$. Then their integral $I(\omega) = \operatorname{Re} J(\omega)$, where

$$J(\omega) = \int_0^\infty dx\, f(x)\, e^{i\omega x}.$$

If now $f(x)$ is analytic in ($\operatorname{Re} x \geqslant 0, \operatorname{Im} x \geqslant 0$), and $f(x)\, e^{i\omega x} = o(x^{-1})$ for $|x| \to \infty$, $\operatorname{Re} x \geqslant 0$, then the path of integration can be rotated into the upper half plane to give

$$I(\omega) = \operatorname{Re} \left\{ i \int_0^\infty dy\, e^{-\omega y} f(iy) \right\},$$

which will often be much more easily evaluated. Similarly, a Bessel function may be replaced by a Hankel function $H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$ which decreases exponentially in the upper half-plane. If desired one may use the complex conjugate functions in (1) and deform the contour into the lower half-plane.

The conditions I have stated above are stronger than necessary. However the analysis is elementary and familiar, and in any particular application it is usually easy to see if a desired path deformation is permissible.

I evaluated the integrals $I_1 - I_{11}$ of Ref. 1 using this method. In most cases the integral was broken into two parts, $(0, \epsilon)$ and (ϵ, ∞) for some finite ϵ. This was done to avoid singularities on the imaginary axis (including the origin). Gaussian integration was used on the $(0, \epsilon)$ portion. On the other portion the change of variables $x = \epsilon + \alpha r e^{i\theta}$ was used to transform the integral to the form $\int_0^\infty dr\, e^{-r} g(r)$, and this was evaluated by Gauss–Laguerre integration. The Gauss-trigonometric quadrature used in Ref. 1 would have been more economical of computer time on the $(0, \epsilon)$ part, but even without that there was no difficulty in computing the integrals to 9 significant

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figures using fewer function evaluations than were used in Ref. 1. The improvement was greater where the methods of Ref. 1 required the most evaluations. Note that after contour rotation all evaluations of f are given positive weight, but that in integration along the real axis half the evaluations receive positive weight and half negative—obviously an awkward way to evaluate an integral.

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Reference

1. M. Blakemore, G. A. Evans, and J. Hyslop, *J. Computational Phys.* **22** (1976), 352.

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